

# An alternative approach to the quasi-Periodic solutions of the Hunter-Saxton hierarchy

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## Abstract

This paper is dedicated to provide the global solutions of algebro-geometric type for all the equations of a new commuting hierarchy containing the Hunter-Saxton (HS) equation. Our main tools include the zero curvature method to derive the HS hierarchy, the generalized Jacobian variety, the generalized Riemann theta function, the Weyl  $m$ -fuctions  $m_{\pm}(x, t, z)$ , and the pole motion obtained by solving an inverse problem for the Sturm-Liouville equation  $L(\psi_1) = -\psi_1'' = zy\psi_1$ . Based on these tools and the theory of nonautonomous differential systems, topological dynamics and ergodic theory, the algebro-geometric solutions are obtained for the entire HS hierarchy.

## 1 Introduction

We study here the algebro-geometric solutions for all the equations of a new commuting hierarchy containing the Hunter-Saxton (HS) equation,

$$u_{xxt} = -uu_{xxx} - 2u_xu_{xx}, \quad (1.1)$$

where  $u(x, t)$  is the function of spatial variable  $x$  and time variable  $t$ . It arises in two different physical contexts in two nonequivalent variational forms [1, 2]. The first is shown to describe the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field [1, 2]. The second is shown to describe the high frequency limit of the Camassa-Holm (CH) equation [10, 11, 32]

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.2)$$

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which was originally introduced in [10, 11] as model equation for shallow water waves, and obtained independently in [31] with a bi-hamiltonian structure.

The HS equation is a completely integrable system with a bi-hamiltonian structure and hence it possesses a Lax pair, an infinite family of commuting Hamiltonian flows, as well as an associated sequence of conservation laws, (Hunter and Zheng [2], Reyes [20]). Traveling waves, asymptotic and piecewise smooth solutions of (1.1) were described in Alber *et al* [7, 8, 9]. The inverse scattering solutions have been obtained by Beals, Sattinger and Szmigielski [19]. Recently, Lenells [23], [24] and also Khesin and Misiólek [22] pointed out that it describes the geodesic flow on the homogeneous space related to the Virasoro group. Bressan and Constantin [25], also Holden [26] constructed a continuous semigroup of weak, dissipative solutions. Yin [27] proved the local existence of strong solutions of the periodic HS equation and showed that all strong solutions-except space independent solutions-blow up in finite time. Gui, Liu and Zhu [28] studied the wave-breaking phenomena and global existence. Furthermore, Morozov [29], Sakovich [30] and Reyes [20], [21] investigated (1.1) from a geometric perspective. In [16], we constructed algebro-geometric solutions of the whole HS hierarchy by using polynomial recursive and spectral analysis method, due to Gesztesy [12].

Quasi-periodic solutions (also called algebro-geometric solutions or finite gap solutions) of nonlinear equations were originally studied on the KdV equation based on the inverse spectral theory and algebro-geometric method developed by pioneers such as the authors in Refs.[3]-[9], then in [12, 33]. Roughly speaking, the algebro-geometric approach consists of finding solutions of HS hierarchy, which are strictly connected to meromorphic functions defined on a Riemann surface, in the sense that the zeros and the poles of such functions completely determine the solution one is looking for. On the other hand, the algebro-geometric setting is directly linked to a linear differential operator of Sturm-Liouville type, in particular to its spectrum. In fact, it is well known that if  $u(x, t)$  is a solution of HS equation, then there exists a positive density function  $y(x, t) = u_{xx}/2$  such that the spectrum of the related Sturm-Liouville operator

$$L\psi_1 = \frac{d^2}{dx^2}\psi_1 = -zy\psi_1$$

does not depend on  $t$  [18]. The study of isospectral classes of  $L$  leads to the determination of the constants of motion for the associated solution  $u$  of HS hierarchy.

Motivated by Zampogni's work [14], in this paper we derive another hierarchy for HS, linked to the Sturm-Liouville operator  $L$ , by using a zero curvature method. This approach has a significant difference from our paper [16]: the algebro-geometric setting of this paper is derived from the spectral properties of  $L$ , and in particular from the properties of the Weyl  $m$ -functions. So we give more relevance to the spectral problem than to the associated zero curvature relation, i.e. the constants of motion derive directly from the isospectral classes of  $L$  and not from the zero-curvature relation. At the same time, the study of the spectral properties of  $L$  is carried out using methods of nonautonomous differential equations and the classical theory of algebraic curves.

We pose the problem of finding all the densities  $y(x)$  such that Hypotheses 2.4 hold. It turns out that this problem is equivalent to that of finding solutions  $u(x, t)$  of the  $r$ -th order HS equation having as initial condition a solution of the stationary  $n$ -th order HS equation. Indeed, there is a surprising relation between the Weyl  $m$ -function of  $L$  and the entries of the matrix satisfying the zero-curvature relation, and this fact enables us to find explicit algebro-geometric solutions for HS hierarchy, when the initial data lie in an isospectral class of  $L$ .

The outline of this paper is as follows. In section 2, we review some recent developments in the theory of the inverse Sturm-Liouville problem. The material discussed here can be found in [15, 38]. We use methods of nonautonomous differential systems, topological dynamics and ergodic theory, to characterize all “ergodic potentials  $p, q, y : \mathbb{R} \rightarrow \mathbb{R}$ ” with  $p, y > 0$ , which constitute a non autonomous differential system (2.1). Such a characterization is carried out by considering the finite poles  $P_1(x), \dots, P_n(x)$  of the Weyl  $m$ -functions  $m_{\pm}(z, x)$ , when interpreted in a “dynamical way”.

In section 3 and section 4, we derive the stationary HS hierarchy and the time-dependent HS hierarchy by using a zero curvature method, respectively. A solution of the  $n$ -th order stationary HS equation is the suitable initial condition for solving the  $r$ -th order time-dependent HS equation, for every fixed  $n > r \in \mathbb{N}$ .

In section 5, we investigate the relation between the matrix  $B_n$  of the stationary formalism and the Weyl  $m$ -functions  $m_{\pm}$  of the Sturm-Liouville system. This relation is the key to solving the HS equation. In particular, we can see that the entry  $F_n$  of  $B_n$  completely determines the Weyl  $m$ -functions  $m_{\pm}$ , or better, their common meromorphic extension  $M(P, x)$  to a Riemann surface  $\mathcal{R}$ . The zeros of  $F_n$  (when viewed as a polynomial of degree  $n$  in  $z$ ) are exactly the poles  $P_i(x)$  ( $i = 1, \dots, n$ ) of the function  $M(P, x)$ . Based on this fact and the results of section 2, we give the expression for the solution

of the  $n$ -th order stationary HS equation in (5.14).

In section 6, we devoted to the time-dependent formalism. We obtain the expression for the solution  $u(x, t_r)$  of the  $r$ -th order HS equation in (6.44). This solution is of class  $C^\infty(\mathbb{R}^2)$ , since the poles  $P_i(x, t)$  have the same regularity properties. Moreover, we study the properties of the Weyl  $m$ -functions  $m_\pm$ . A Riccati type equation with respect to the time variable  $t$  is obtained.

In section 7, we investigate the  $(x, t)$ -motion of the poles  $P_i(x, t)$ . It turns out that such a motion can be made clear by considering its isomorphic image through a generalized Abel map on the generalized Jacobian variety of a Riemann surface  $\mathcal{R}$  of genus  $n$ . In particular, we will find that the motion on the generalized Jacobian variety can be described by using a basis  $(d\omega_1, \dots, d\omega_n)$  of holomorphic differentials on  $\mathcal{R}$ , plus a non-holomorphic differential  $d\omega_0$ , in such a way that the  $x$ -motion is confined to the non-holomorphic coordinate, and defines there a linear function  $\omega_0(x)$ , while the  $t$ -motion determines a triangular structure on the first  $n - 1$  of the holomorphic coordinates. The image of the  $t$ -motion through the generalized Abel map is determined by functions  $\omega_i(x, t)$  ( $i = 1, \dots, n - 1$ ). The last holomorphic coordinate  $\omega_n(x, t)$  remains implicitly defined in such a way that the vector  $(\omega_0(x), \omega_1(x, t), \dots, \omega_n(x, t))$  is contained in a translate  $\Upsilon_0$  of the zero locus of the generalized Riemann theta function  $\Theta_0(z)$ , for every  $x, t \in \mathbb{R}$ . In addition, we give a brief description of how we can build singular Riemann surfaces (called “generalized Riemann surfaces”), and explain how the classical theory can be extended to these new objects.

In section 8, we show that the elementary symmetric functions of  $n$ -tuples of distinct points in  $\mathcal{R}$  can be expressed in terms of a generalized Riemann theta function. It turns out that the  $j$ -th symmetric function of points  $P_1, \dots, P_n \in \mathcal{R}$ , depends on the partial derivative with respect to the coordinate  $\omega_{n-j}$  of the logarithm of  $w_0(P_1(x, t_r), \dots, P_n(x, t_r))$ . As an application, we derive the Riemann theta function representation for the solution  $u(x, t)$  of the HS hierarchy.

## 2 Preliminaries

In this section, we give some important terminology and results about the Sturm-Liouville operator, in particular a dichotomy-theoretic approach to the inversion problem for such operator. A complete treatment of the results in this section can be found in [15, 38].

Let  $A$  be a compact metric space and  $\{\tau_x, x \in \mathbb{R}\}$  a family of homeo-

morphisms of  $A$  such that

- (i)  $\tau_0(a) = a$ , for all  $a \in A$ ;
- (ii)  $\tau_{x+s}(a) = \tau_x \circ \tau_s(a)$ , for every  $a \in A$  and  $x, s \in \mathbb{R}$ ;
- (iii) the map  $\tau : A \times \mathbb{R} \rightarrow A : (a, x) \mapsto \tau_x(a)$  is continuous.

A family  $\{\tau_x\}$  satisfying (i)-(iii) is called a *flow* on  $A$ . We fix a  $\{\tau_x\}$ -ergodic measure  $\mu$  on  $A$ , and assume that  $A$  is the topological support of  $\mu$ , i.e.  $\mu(V) > 0$  for every open set  $V \subset A$ . The triple  $(A, \{\tau_x\}, \mu)$  is called a *stationary ergodic process*.

Let  $p, q, y : A \rightarrow \mathbb{R}$  be continuous functions with  $p, y$  strictly positive; for every  $a \in A$ , we consider the map (denoted by  $a(x)$ , with abuse of notation),

$$x \mapsto \begin{pmatrix} 0 & 1/p(\tau_x(a)) \\ q(\tau_x(a)) - zy(\tau_x(a)) & 0 \end{pmatrix}.$$

We study the family of differential equations

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1/p(\tau_x(a)) \\ q(\tau_x(a)) - zy(\tau_x(a)) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad a \in A, z \in \mathbb{C}, \quad (2.1)$$

which is equivalent to  $-(p\psi_1')' + q\psi_1 = zy\psi_1$ , if the map  $a \mapsto \frac{d}{dx}p(\tau_x(a))|_{x=0}$  is well defined and continuous.

Let  $\Phi_a(x)$  be the fundamental matrix solution for (2.1).

**Definition 2.1** The family of equations (2.1) is said to have an *exponential dichotomy* over  $A$ , if there are positive constants  $K, \gamma$  and a continuous function  $a \mapsto P_a : A \rightarrow \mathcal{P} = \{\text{linear projections } P : \mathbb{C}^2 \rightarrow \mathbb{C}^2\}$  such that the following estimates hold:

- (i)  $\|\Phi_a(x)P_a\Phi_a(s)^{-1}\| \leq Ke^{-\gamma(x-s)}, \quad x \geq s,$
- (ii)  $\|\Phi_a(x)(I - P_a)\Phi_a(s)^{-1}\| \leq Ke^{\gamma(x-s)}, \quad x \leq s.$

It follows from the fact that  $\text{tr}(a(x))=0$  for all  $x \in \mathbb{R}$  that both the image  $\text{Im}P_a$  and the kernel  $\text{Ker}P_a$  are one-dimensional, i.e., can be viewed as complex lines in  $\mathbb{C}^2$ . The following proposition shows an important characterization of the spectrum of (2.1) for  $\mu$ -a.e.  $a \in A$ . A proof can be found in [15].

**Proposition 2.2** *For  $\mu$ -a.e.  $a \in A$ , the spectrum  $\Sigma_a$  of (2.1) equals a closed set  $\Sigma \subset \mathbb{R}$  which does not depend on the choice of  $a$ . Moreover,*

$$\mathbb{C} \setminus \Sigma = \{z \in \mathbb{C}, (2.1) \text{ has an exponential dichotomy}\}.$$

Let now  $a \in A$  and  $z \in \mathbb{C} \setminus \Sigma$ . We define  $m_{\pm}(a, z)$  to be the unique complex numbers such that  $\text{Im } P_a = \text{Span}(1, m_+(a, z))^T$  and  $\text{Ker } P_a = \text{Span}(1, m_-(a, z))^T$ . As usual, we set  $m_+ = \infty$  (or  $m_- = \infty$ ) if  $\text{Im } P_a = \text{Span}(0, 1)^T$  (or  $\text{Ker } P_a = \text{Span}(0, 1)^T$ ). Moreover, it is easily proved that  $\Im m_+ \Im z > 0$  and  $\Im m_- \Im z < 0$  for every  $z$  with  $\Im z \neq 0$ . For every fixed  $a \in A$ , the maps  $z \mapsto m_{\pm}(a, z)$  are analytic in  $\mathbb{C} \setminus \mathbb{R}$ , and are meromorphic in  $\mathbb{C} \setminus \Sigma$ . They coincide with the classical Weyl  $m$ -functions (the definition and the properties of the classical Weyl  $m$ -functions, see [35, 36]). For every  $z \in \mathbb{C}$ , let  $\beta(z)$  be the upper Lyapunov exponent for (2.1).

**Proposition 2.3** *Fix  $a \in A$ . Let  $I \subset \mathbb{R}$  be an open interval and suppose that  $\beta(z) = 0$  for a.e.  $z \in I$ . Then  $m_{\pm}(a, z)$  extend holomorphically through  $I$ . If  $h_{\pm}$  denote the extensions of  $m_{\pm}$ , then*

$$h_+(z) = \begin{cases} m_+(a, z), & \Im z > 0 \\ m_-(a, z), & \Im z < 0 \end{cases} \quad \text{and} \quad h_-(z) = \begin{cases} m_-(a, z), & \Im z > 0 \\ m_+(a, z), & \Im z < 0 \end{cases}.$$

A proof of this proposition can be found in [15].

Throughout all the paper we make the following fundamental assumptions.

**Hypotheses 2.4** (H1) *The spectrum  $\Sigma$  of (2.1) is a finite union of intervals, i.e.  $\Sigma = [z_2, z_3] \cup \dots \cup [z_{2n}, \infty)$ ;*

(H2) *The Lyapunov exponent  $\beta(z)$  vanishes a.e. in the spectrum, i.e.  $\beta(z) = 0$  for a.e.  $z \in \Sigma$ .*

Now we give the algebro-geometric structure of this paper. The results stated following are proved in [38]. We first look for formula of algebro-geometric type for the functions  $p, q$  and  $y$ . Consider the Riemann surface  $\mathcal{R}$  described by the algebraic relation

$$w^2 = z \prod_{m=1}^{2n} (z - z_m).$$

As usual,  $\mathcal{R}$  is obtained by the union of two Riemann spheres  $\mathbb{C}^2 \cup \{\infty\}$  cut open along  $\Sigma$  and glued together in the usual way, with genus  $n$  and exactly

$2n + 2$  ramification points, namely  $0, z_1, \dots, z_{2n}, \infty$ . Let  $\pi : \mathcal{R} \rightarrow \mathbb{C} \cup \{\infty\}$  be the canonical projection and  $k(P)$  be the meromorphic function on  $\mathcal{R}$  defined by

$$k(z) = \sqrt{(z - z_1) \dots (z - z_{2n})}, \quad \pi(P) = z.$$

Then both  $m_+(a, \cdot)$  and  $m_-(a, \cdot)$  define a single meromorphic function  $M_a(\cdot)$  on  $\mathcal{R}$ , such that  $M_a = m_+$  and  $M_a \circ \sigma = m_-$ , where  $\sigma$  is the map interchanging the sheets.  $M_a(P)$  satisfies the Riccati equation

$$M_x + z^{1/2} \frac{1}{p} M^2 = z^{-1/2} q - z^{1/2} y, \quad \pi(P) = z.$$

The function  $M_a$  has exactly  $n$  finite poles  $P_1(x), \dots, P_n(x)$ , one for each interval in the resolvent set  $\mathbb{R} \setminus \Sigma$ . For the properties of the function  $M_a$  and a complete discussion about its poles and the algebro-geometric setting, see [38].

If we put  $T(z) = \sum_{i=1}^n (z - \pi(P_i(x)))$ , then we have

$$\begin{aligned} M_a(P) = m_+(a, P) &= \frac{Q(z) + \sqrt{py}k(P)}{T(z)}, \quad \pi(P) = z \\ M_a \circ \sigma(P) = m_-(a, P) &= \frac{Q(z) - \sqrt{py}k(P)}{T(z)}, \quad \pi(P) = z, \end{aligned} \quad (2.2)$$

where  $Q(z) = \frac{i(py)'}{4y} z^{n-1/2} + q_{n-1} z^{n-3/2} + \dots + q_0(x)$ , and  $Q(\pi(P_i)) = \sqrt{py}k(P_i)$ , for every  $i = 1, \dots, n$ .

The function  $M_a$  can be viewed as a function of the variable  $x$  by letting  $a \in A$ , and considering the map  $(x, P) \mapsto M_{\tau_x(a)}(P)$ . The same holds for  $p, q, y$ . It follows that the poles of  $M_{\tau_x(a)}(P)$  depend on  $x$ , hence we obtain the “moving poles”  $P_1(x), \dots, P_n(x)$ . In addition, it is clear that  $P_1(0) = P_1, \dots, P_n(0) = P_n$  are the finite poles of the function  $M_a(P)$ .

Based on this algebro-geometric structure, we are able to find formulas for the functions  $p(x), q(x), y(x)$  in terms of the finite poles and the zeros of  $M_{\tau_x(a)}(P)$ , for every  $a \in A$ . For our convenience, we write  $P_i(x)$  instead of  $\pi(P_i(x))$ , if no confusion arises.

**Theorem 2.5** *Suppose that Hypotheses 2.4 hold. Then for every  $a \in A$ , the finite poles  $P_j(x)$  ( $j = 1, \dots, n$ ) of the function  $M_{\tau_x(a)}(P)$  satisfy the following first-order system of differential equations*

$$P_{j,x}(x) = \frac{(-1)^n k(P_j(x)) \sqrt{P_j(x)} [m_-^0(x) - m_+^0(x)] \prod_{i=1}^n P_i(x)}{p(x) k(0) \prod_{s \neq j} (P_j(x) - P_s(x))}, \quad (2.3)$$

where  $m_{\pm}^0(x) := m_{\pm}(\tau_x(a), 0)$ . Moreover, the functions  $p(x), q(x), y(x)$  satisfy the following relations involving the finite poles of the meromorphic function  $M_{\tau_x(a)}(P)$  on  $\mathcal{R}$ ,

$$\frac{2\sqrt{py}}{p} = \frac{T_x(P_j(x))}{k(P_j(x))\sqrt{P_j(x)}} = \frac{(-1)^{n+1}[m_{-}^0(x) - m_{+}^0(x)] \prod_{i=1}^n P_i(x)}{p(x)k(0)},$$

and

$$q(x) = y(x) \left( \sum_{j=1}^{2n} z_j - 2 \sum_{i=1}^n P_i(x) \right) + q_{n,x}(x) + \frac{q_n^2(x)}{p(x)},$$

where

$$\begin{aligned} q_n(x) &= \frac{i(py)'}{4y} = \frac{ip(x)}{2} \frac{d}{dx} \left( \ln \left( [m_{-}^0(x) - m_{+}^0(x)] \prod_{i=1}^n P_i(x) \right) \right) \\ &= \sum_{j=1}^n \frac{\sqrt{p(x)y(x)k(P_j(x))}}{\sqrt{P_j(x)} \prod_{r \neq j} (P_j(x) - P_r(x))} - \sum_{j=1}^n \frac{Q(0, x)}{\sqrt{P_j(x)} \prod_{r \neq j} (P_j(x) - P_r(x))}. \end{aligned}$$

The proof of this theorem can be found in [38].

What is of great relevance is the fact that this problem can be inverted, i.e. the following holds.

**Theorem 2.6** *Let  $z_1 < z_2 < \dots < z_{2n}$  be distinct positive real numbers, and let  $\mathcal{R}$  be the Riemann surface genus of  $n$  described by the following algebraic relation*

$$w^2 = z(z - z_1)(z - z_2) \cdots (z - z_{2n}).$$

*Let  $\tilde{P}_1, \dots, \tilde{P}_n$  be points on  $\mathcal{R}$  such that  $z_{2j-1} \leq \tilde{P}_j \leq z_{2j}$  ( $1 \leq j \leq n$ ). Then there exists a stationary ergodic process  $(A, \{\tau_x\}, \mu)$  together with functions  $p, q, y : A \rightarrow \mathbb{R}$ , such that the spectrum of the family*

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1/p(\tau_x(a)) \\ q(\tau_x(a)) - zy(\tau_x(a)) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad a \in A, z \in \mathbb{C}, \quad (2.4)$$

*of differential equations does not depend on the choice of  $a \in A$  and has the form*

$$\Sigma = [z_2, z_3] \cup [z_4, z_5] \cup \dots \cup [z_{2n}, \infty). \quad (2.5)$$

*Moreover,  $\beta(z) = 0$ , for Lebesgue a.e.  $z \in \Sigma$ . The functions  $p, q, y$  satisfy relations analogous to those of Theorem 2.5.*



The proof of this theorem can be found in [38]. The importance of Theorem 2.6 lies in the fact that we can generate infinitely many stationary ergodic processes and functions  $p, q, y$  such that the above conclusions hold. For more explanations about this importance, see again [38].

### 3 The stationary HS hierarchy

In this section, we derive the stationary HS hierarchy and the corresponding sequence of zero-curvature pairs by using a polynomial recursion formalism. We will use the ergodic-dynamical structure of Section 2 in the following.

Let  $\mathcal{L}$  be the set of all the positive uniformly continuous bounded functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We equip  $\mathcal{L}$  with the compact open topology. Let  $\{\tau_x\}$  be the *Bebutov flow* on  $\mathcal{L}$ , that is,  $\tau_x(f)(\cdot) = f(x + \cdot)$ , for every  $f \in \mathcal{L}$ . Moreover, we define  $\mathcal{A} = \text{Hull}(y_0) = \text{cls}\{\tau_x(y_0), x \in \mathbb{R}, y_0 \in \mathcal{L}\}$ . One infers that  $\mathcal{A}$  is compact. For every  $A \in \mathcal{A}$ , let  $y(A) = A(0)$ . We consider the family of equations

$$X' = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -zy(\tau_x(A)) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = A(x)X, \quad A \in \mathcal{A}, \quad z \in \mathbb{C}, \quad (3.1)$$

which is equivalent to  $-\psi_1'' = zy\psi_1$ .

Now we start with the following  $2 \times 2$  matrix isospectral problem

$$\psi_x = A(x, z)\psi \quad (3.2)$$

and an auxiliary problem

$$\psi_{t_n} = B_n(z)\psi, \quad (3.3)$$

where  $A(x, z)$  is a Sturm-Liouville matrix, defined by

$$A(x, z) = \begin{pmatrix} 0 & 1 \\ -zy(x) & 0 \end{pmatrix} \quad (3.4)$$

and  $B_n(z)$  is a two dimensional matrix-valued linear differential operator, defined by

$$B_n(z) = \begin{pmatrix} -G_n(z) & z^{-1}F_n(z) \\ -H_n(z) & G_n(z) \end{pmatrix} \quad z \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N}, \quad (3.5)$$

assuming  $F_n$ ,  $G_n$  and  $H_n$  to be polynomials of degree at most  $n$  with  $C^\infty$  coefficients with respect to  $x$ . The compatibility condition between (3.2) and (3.3) yields the stationary zero-curvature equation

$$-B_{n,x} + [A, B_n] = 0, \quad (3.6)$$

namely

$$F_{n,x} = 2zG_n, \quad (3.7)$$

$$H_{n,x} = -2zyG_n, \quad (3.8)$$

$$G_{n,x} = H_n - yF_n. \quad (3.9)$$

From (3.7)-(3.9), a direct calculation shows that

$$\frac{d}{dx} \det(B_n(z, x)) = -\frac{1}{z} \frac{d}{dx} \left( zG_n(z, x)^2 - F_n(z, x)H_n(z, x) \right) = 0. \quad (3.10)$$

Hence,  $zG_n^2 - F_nH_n$  is  $x$ -independent implying

$$zG_n^2 - F_nH_n = k^2(z), \quad (3.11)$$

where the integration constant  $k^2(z)$  is a polynomial of degree  $2n$  with respect to  $z$ . If  $\{z_m\}_{m=1, \dots, 2n}$  denote its zeros, then

$$k^2(z) = \prod_{m=1}^{2n} (z - z_m), \quad \{z_m\}_{m=1, \dots, 2n} \in \mathbb{C}. \quad (3.12)$$

Next, we introduce the following polynomial  $F_n(z)$ ,  $G_n(z)$  and  $H_n(z)$  with respect to the spectral parameter  $z$ ,

$$F_n(z) = \sum_{l=0}^n f_l z^l, \quad (3.13)$$

$$G_n(z) = \sum_{l=0}^{n-1} g_l z^l, \quad (3.14)$$

$$H_n(z) = \sum_{l=0}^n h_l z^l. \quad (3.15)$$

From (3.7), (3.8) and (3.11), we obtain that  $f_0, h_0$  is a constant and  $f_0 h_0 = -\prod z_m$ . For our convenience in the following, we choose  $f_0$  such that

$$f_0 = -h_0 = -\sqrt{\prod z_m}. \quad (3.16)$$

Without loss of generality, let  $f_0 = h_0 = 1$  to normalize the polynomials  $F_n$  and  $H_n$ . We keep the same notation for the normalized polynomials, unless explicitly stated.

We begin our manipulation.

First, substituting (3.7) into (3.9) and (3.8), we arrive at

$$\frac{1}{2}F_{n,xx} - zH_n + zyF_n = 0, \quad (3.17)$$

$$H_{n,x} + yF_{n,x} = 0. \quad (3.18)$$

Then, differentiating (3.17) with respect to  $x$ , we obtain

$$\frac{1}{2}F_{n,xxx} = zH_{n,x} - zy_xF_n - zyF_{n,x}. \quad (3.19)$$

By subtraction and addition of (3.19) to (3.18), we have

$$\frac{1}{2}F_{n,xxx} = -2zyF_{n,x} - zy_xF_n, \quad (3.20)$$

$$\frac{1}{2}F_{n,xxx} = 2zH_{n,x} - zy_xF_n. \quad (3.21)$$

Hence, comparing the coefficients of the same powers in (3.20), we have the following recursion formalism for the coefficients  $f_l$ :

$$f_{l,x} = -\mathcal{G}(4yf_{l-1,x} + 2y_xf_{l-1}), \quad (3.22)$$

where  $\mathcal{G}$  is given by

$$(\mathcal{G}v)(x) = \int_{-\infty}^x \int_{-\infty}^{x_1} v(x_2) dx_2 dx_1, \quad x \in \mathbb{R}, v \in L^\infty(\mathbb{R}). \quad (3.23)$$

It is easy to see that  $\mathcal{G}$  is the resolvent of the one-dimensional Laplacian operator, that is

$$\mathcal{G} = \left( \frac{d^2}{dx^2} \right)^{-1}. \quad (3.24)$$

Explicitly, we compute

$$\begin{aligned} f_0 &= 1, \\ f_1 &= -u + c_1, \\ f_2 &= \mathcal{G}(u_{xx}u + \frac{1}{2}u_x^2) - uc_1 + c_2, \\ &\text{etc.} \end{aligned} \quad (3.25)$$

where  $\{c_l\}_{l \in \mathbb{N}} \subset \mathbb{C}$  are integration constants and we have used the assumption  $f_l(u)|_{u=0} = c_l$ ,  $l \in \mathbb{N}$ .

The relation (3.7) and (3.21) provide the recursion formalism for the coefficients  $g_l$  and  $h_l$  of the polynomial  $G_n$  and  $H_n$ , one can refer to [16].

For fixed  $n \in \mathbb{N}$ , by using (3.9), for  $l = n$  we obtain  $h_n = yf_n$ , and together with (3.21) yields,

$$\text{s-HS}_n(u) = 2u_{xx}f_{n,x}(u) + u_{xxx}f_n(u) = 0. \quad (3.26)$$

This is the stationary HS equation of order  $n$ .

Hence, we obtain the stationary HS hierarchy. The first equation in the hierarchy, obtained by taking  $n = 1$ , we compute explicitly,

$$\text{s-HS}_1(u) = -2u_{xx}u_x - u_{xxx}u + u_{xxx}c_1 = 0. \quad (3.27)$$

It is easy to see that  $\text{s-HS}_1(u) = 0$  ( $c_1 = 0$ ) represents the classical one dimensional stationary HS equation.

## 4 The time-dependent HS hierarchy

In this section, we will introduce the time-dependent HS hierarchy. This means that  $u$  are now considered as functions of both space and time. Let  $r \in \mathbb{N}$  be fixed. We introduce a deformation parameter  $t_r \in \mathbb{R}$  in  $u$ , replacing  $u(x)$  by  $u(x, t_r)$ , for each equation in the hierarchy.

Now we consider the Sturm-Liouville matrix

$$A = \begin{pmatrix} 0 & 1 \\ -zy(x, t_r) & 0 \end{pmatrix} \quad (4.1)$$

and the matrix  $B_r(x, t_r)$  whose entries are polynomials  $F_r$  and  $H_r$  of degree  $r$  in  $z$ ,  $G_r$  of degree  $r - 1$  in  $z$ , and with coefficients depending on  $t_r$  as well. Then the compatibility condition yields the zero-curvature equation

$$A_{t_r} - B_{r,x} + [A, B_r] = 0, \quad r \in \mathbb{N}, \quad (4.2)$$

namely

$$-zy_{t_r} + H_{r,x} + 2zyG_r = 0, \quad (4.3)$$

$$F_{r,x} = 2zG_r, \quad (4.4)$$

$$G_{r,x} = H_r - yF_r. \quad (4.5)$$

Substituting (4.4) into (4.5) and (4.3), we get

$$\frac{1}{2}F_{r,xx} = zH_r - zyF_r, \quad (4.6)$$

$$-zy_{t_r} + H_{r,x} + yF_{r,x} = 0. \quad (4.7)$$

Then as in the stationary case, differentiating (4.6) with respect to  $x$ , and by addition and subtraction to (4.7), we arrive at

$$\frac{1}{2}F_{r,xxx} = z^2y_{t_r} - 2zyF_{r,x} - zy_xF_r, \quad (4.8)$$

$$\frac{1}{2}F_{r,xxx} = -z^2y_{t_r} + 2zH_{r,x} - zy_xF_r. \quad (4.9)$$

Hence, the relation (4.8) gives the recursion formalism for the coefficients  $f_l$ :

$$\begin{aligned} f_1 &= -u + c_1, \\ \frac{1}{2}f_{2,xxx} &= y_{t_r} - 2yf_{1,x} - y_xf_1, \\ f_{l,x} &= -\mathcal{G}(4yf_{l-1,x} + 2y_xf_{l-1} - 2\delta_{l,2}y_{t_r}). \end{aligned} \quad (4.10)$$

From (4.10), we infer that all the higher order coefficients depend in an implicit way on  $t_r$ , since  $f_2$  depends on  $t_r$ .

Moreover, the coefficients  $h_l$  of  $H_r$  are determined by (4.9). Since from (4.5), we obtain  $h_r = yf_r$ , then together with (4.9) yields

$$\text{HS}_r(u) = 2y(x, t_r)f_{r,x}(x, t_r) + y_x(x, t_r)f_r(x, t_r) = 0, \quad r \geq 2. \quad (4.11)$$

which is the  $r$ -th order HS equation. In addition, in the case  $r = 1$ , the corresponding equation can also be derived from (4.9), that is,

$$\text{HS}_1(u) = u_{xxt_1} + 2u_{xx}u_x + uu_{xxx} - u_{xxx}c_1 = 0. \quad (4.12)$$

It is clear that  $\text{HS}_1(u) = 0$  ( $c_1 = 0$ ) represents the classical HS equation.

## 5 The stationary HS formalism

In this section we focus our attention on the stationary case. By solving the inverse Sturm-Liouville problem, we obtain the relation between the Weyl  $m$ -functions  $m_{\pm}$  and the entries of the matrix  $B_n$  of the stationary HS hierarchy, which will bring to the determination of the solution of the HS hierarchy both in stationary and time-dependent cases.

Consider the polynomials  $F_n, G_n$  and  $H_n$  as defined before in the stationary case, then (3.11) gives,

$$zG_n^2 - F_nH_n = k^2(z) = \prod_{m=1}^{2n} (z - z_m). \quad (5.1)$$

We introduce the hyperelliptic curve  $\mathcal{K}_n$ ,

$$\mathcal{K}_n : w^2 - zk^2(z) = w^2 - z \prod_{m=1}^{2n} (z - z_m) = 0, \quad \{z_m\}_{m=1, \dots, 2n} \in \mathbb{C}, \quad (5.2)$$

which is compactified by joining the point  $P_\infty$  at infinity. The complex structure on  $\mathcal{K}_n$  is defined in the usual way [12]. Hence,  $\mathcal{K}_n$  becomes a two-sheeted hyperelliptic Riemann surface of genus  $n$  (possibly with a singular affine part) in a standard manner, denoted by  $\mathcal{R}$ .

Let  $\mu_j, \nu_j : \mathbb{R} \rightarrow \mathbb{R}$  ( $j = 1, \dots, n$ ) be continuous functions with  $\mu_i \neq \mu_j$ ,  $\nu_i \neq \nu_j$  for  $i \neq j$ , and such that

$$F_n(z) = \frac{(-1)^{n+1}k(0)}{\prod_{j=1}^n \mu_j(x)} \prod_{j=1}^n (z - \mu_j(x)) \quad (5.3)$$

and

$$H_n(z) = \frac{(-1)^n k(0)}{\prod_{j=1}^n \nu_j(x)} \prod_{j=1}^n (z - \nu_j(x)). \quad (5.4)$$

Next, we define the function  $\tilde{M} : \mathcal{R} \times \mathbb{R} \rightarrow \mathbb{C}$ ,

$$\tilde{M}(P, x) = \frac{k(P) + \sqrt{z}G_n(z)}{F_n(z)} = \frac{-H_n(z)}{k(P) - \sqrt{z}G_n(z)}, \quad \pi(P) = z. \quad (5.5)$$

**Theorem 5.1** *For every fixed  $P \in \mathcal{R}$  such that  $\pi(P) = z \in \mathbb{C} \setminus \Sigma$ ,  $\tilde{M}(P, x)$  satisfies the following Riccati equation*

$$\tilde{M}_x(P, x) + z^{1/2}\tilde{M}(P, x)^2 = -z^{1/2}y. \quad (5.6)$$

**Proof.** Using (5.5), (3.7), (3.9) and (3.11), a direct calculation shows that (5.6) is true.  $\square$

The following theorem describes the nature of the  $x$ -motion of the zeros  $\{\mu_j(x)\}_{j=1, \dots, n}$  of  $F_n$ .

**Theorem 5.2** *Assume that the zeros  $\{\mu_j(x)\}_{j=1,\dots,n}$  of  $F_n$  remain distinct, then  $\{\mu_j(x)\}_{j=1,\dots,n}$  satisfy the system of differential equations,*

$$\mu_{j,x}(x) = \frac{2(-1)^n k(\mu_j(x)) \sqrt{\mu_j(x)} \prod_{i=1}^n \mu_i(x)}{k(0) \prod_{\substack{i=1 \\ i \neq j}}^n (\mu_j(x) - \mu_i(x))}. \quad (5.7)$$

Moreover, given initial data satisfying  $\mu_j(x_0) \in [z_{2j-1}, z_{2j}]$  ( $j = 1, \dots, n$ ), then

$$\mu_j(x) \in [z_{2j-1}, z_{2j}], \quad j = 1, \dots, n, \quad x \in \mathbb{R}. \quad (5.8)$$

In particular,  $\mu_j(x)$  changes sheets whenever it hits  $z_{2j-1}$  or  $z_{2j}$ .

**Proof.** The derivatives of (5.3) with respect to  $x$  take on

$$F_{n,x}(\mu_j) = \frac{(-1)^n k(0)}{\prod \mu_j} \mu_{j,x} \prod_{\substack{i=1 \\ i \neq j}}^n (\mu_j - \mu_i). \quad (5.9)$$

On the other hand, inserting  $z = \mu_j$  into (3.7) yields

$$F_{n,x}(\mu_j) = 2\mu_j G_n(\mu_j) = 2\sqrt{\mu_j} k(\mu_j). \quad (5.10)$$

Comparing (5.9) and (5.10) leads to (5.7). To see (5.8), one can use the local coordinate  $\lambda = (z - z_*)^{1/2}$  near a ramification point  $z_*$  [12, 16].  $\square$

Now we explain the importance of the relation (5.7) from two aspects. First, it shows the behavior of a point  $\mu_j(x)$  when it reaches a ramification point of the resolvent interval containing it. In fact, as soon as the point  $\mu_j(x)$  reaches a ramification point  $z_*$ , its derivative becomes zero ( $k(z_*) = 0$ ), and it passes from one sheet to the other in the Riemann surface  $\mathcal{R}$ , i.e.,  $\mu_j(x) \in [z_{2j-1}, z_{2j}]$  for every  $j = 1, \dots, n$ . Second, from Theorem 2.6, we know that for every choice of points  $\tilde{P}_j \in [z_{2j-1}, z_{2j}]$  ( $j = 1, \dots, n$ ), there exists a meromorphic function  $P \rightarrow M(x, P)$  on  $\mathcal{R}$ , having as finite simple poles the points  $P_j(x)$  ( $j = 1, \dots, n$ ), satisfying (2.3), which, adapted to the present setting ( $p = 1, q = 0, m_-^0(x) = -m_+^0(x) = 1$ ) becomes

$$P_{j,x}(x) = \frac{2(-1)^n k(P_j(x)) \sqrt{P_j(x)} \prod_{i=1}^n P_i(x)}{k(0) \prod_{s \neq j} (P_j(x) - P_s(x))}. \quad (5.11)$$

Hence, the points  $\mu_j(x)$  satisfy the system (5.11), as well as the poles of the meromorphic function  $M_y$  on  $\mathcal{R}$  defined as in (2.2).

At this point we can use Theorem 2.6 again to prove that there is a function  $y(x)$  satisfying the assumptions on the spectrum (H1) and (H2), and such that the poles  $P_j(x)$  of the function  $M_y$  satisfy  $\pi(P_j(x)) = \mu_j(x)$ , for every  $j = 1, \dots, n$ .

Another important fact from (5.7) is that  $\mu_j(x) \in C^\infty(\mathbb{R})$ , which has been proved in our latest paper [16].

Based on such a  $y(x)$ , obtained by solving the inverse Sturm-Liouville problem, we obtain the following fundamental theorem.

**Theorem 5.3** *Let  $P_j(x)$  ( $j = 1, \dots, n$ ) be  $n$  distinct points on the Riemann surface  $\mathcal{R}$  satisfying (5.11), and such that  $P_j(x_0) = \tilde{P}_j \in [z_{2j-1}, z_{2j}]$  ( $j = 1, \dots, n$ ). Then there exists a positive continuous function  $y(x)$  such that*

$$F_n(z) = \frac{1}{\sqrt{y(x)}} \prod_{j=1}^n (z - \pi(P_j(x))), \quad (5.12)$$

$$M_y(P, x) = \tilde{M}(P, x) = m_+(P, x) \quad \text{and} \quad \tilde{M}(P, x) \circ \sigma(P) = m_-(P, x), \quad (5.13)$$

for every  $x \in \mathbb{R}$  and  $P \in \mathcal{R}$ . Moreover, with such  $y(x)$ , Hypotheses 2.4 hold.

**Remark 5.4** *The similar formulas describing the nature of the  $x$ -motion of the points  $\nu_j(x)$  can be found in [16], here we omit the details. In addition, the regularity properties, together with the other observations concerning the motion which are valid for  $\mu_j(x)$ , also hold for  $\nu_j(x)$ .*

We must emphasize an important difference between our approach in [16] and that in the present paper: while in the former approach, we deduce the algebro-geometric structure directly from the hierarchy, in the latter we start from an assigned algebro-geometric structure and prove that it is conserved by the hierarchy.

Given these preparations, one of the main results of this section, that is, the algebro-geometric formula for the solution  $u(x)$  of the stationary HS hierarchy reads as follows.

**Theorem 5.5** *The solution  $u(x)$  of the stationary HS hierarchy is*

$$u(x) = \sum_{j=1}^n \frac{1}{P_j(x)} - \sum_{m=1}^{2n} \frac{1}{z_m}, \quad (5.14)$$

where the points  $P_j(x)$  ( $j = 1, \dots, n$ ) evolve according to (5.11).



**Proof.** Let  $z = 0$  in (5.5), we obtain

$$m_+^0(x) = -1, \quad m_-^0(x) = +1, \quad (5.15)$$

where we used (3.16) and (5.13). Hence, take  $p = 1$  in Theorem 2.5, we arrive at

$$\sqrt{y(x)} = \frac{(-1)^{n+1}}{k(0)} \prod_{j=1}^n P_j(x). \quad (5.16)$$

Next, we consider the coefficient of  $z$  in (5.12),

$$f_1(x) = \frac{(-1)^{n-1}}{\sqrt{y(x)}} \left( \sum_{j=1}^n \prod_{\substack{i=1 \\ i \neq j}}^n P_i(x) \right). \quad (5.17)$$

Inserting (5.16) into (5.17), we get

$$f_1(x) = \sqrt{\prod_{m=1}^{2n} z_m} \left( \sum_{j=1}^n \frac{1}{P_j(x)} \right). \quad (5.18)$$

On the other hand, we note that the coefficient  $f_1(x)$  in (3.25) is not normalized, we find

$$u(x) - c_1 = \frac{1}{\sqrt{\prod_{m=1}^{2n} z_m}} f_1(x). \quad (5.19)$$

Hence

$$u(x) = \sum_{j=1}^n \frac{1}{P_j(x)} + c_1. \quad (5.20)$$

The constant  $c_1$  can be determined from (3.11), by computing the coefficient of  $z$ , we obtain

$$c_1 = - \sum_{m=1}^{2n} \frac{1}{z_m}. \quad (5.21)$$

## 6 The time-dependent HS formalism

The basic problem in an algebro-geometric construction of the solutions of the time-dependent HS hierarchy is to solve the time-dependent  $r$ -th HS equation with a stationary solution of the  $n$ -th equation as initial data in the hierarchy (recall that  $r < n$ ).

We employ the notations  $\tilde{B}_r, \tilde{F}_r, \tilde{G}_r, \tilde{H}_r, \tilde{f}_r, \tilde{g}_r, \tilde{h}_r$  to stand for the time-dependent quantities, and we will keep the usual notations for the stationary quantities.

Summing up, we are seeking a solution  $u(x, t_r)$  of the time-dependent algebro-geometric initial value problem

$$\text{HS}_r(u) = 2y(x, t_r)\tilde{f}_{r,x}(x, t_r) + y_x(x, t_r)\tilde{f}_r(x, t_r) = 0, \quad (6.1)$$

$$u|_{t_r=t_{0,r}} = u_0,$$

$$\text{s-HS}_n(u_0) = u_{0,xxx}f_n + 2u_{0,xx}f_{n,x} = 0, \quad (6.2)$$

where

$$2y(x, t_r) = u_{xx}(x, t_r). \quad (6.3)$$

We start from the zero-curvature equations:

$$A_{t_r}(x, t_r) - \tilde{B}_{r,x}(x, t_r) + [A(x, t_r), \tilde{B}_r(x, t_r)] = 0, \quad (6.4)$$

$$-B_{n,x}(x, t_r) + [A(x, t_r), B_n(x, t_r)] = 0, \quad (6.5)$$

where

$$\tilde{B}_r(z) = \begin{pmatrix} -\tilde{G}_r(z) & z^{-1}\tilde{F}_r(z) \\ -\tilde{H}_r(z) & \tilde{G}_r(z) \end{pmatrix} \quad (6.6)$$

with entries

$$\tilde{F}_r(z) = \sum_{s=0}^r \tilde{f}_s(x, t_r)z^s, \quad (6.7)$$

$$\tilde{G}_r(z) = \sum_{s=0}^{r-1} \tilde{g}_s(x, t_r)z^s, \quad (6.8)$$

$$\tilde{H}_r(z) = \sum_{s=0}^r \tilde{h}_s(x, t_r)z^s. \quad (6.9)$$

Moreover, it is more convenient for us to rewrite the zero-curvature equations (6.4) and (6.5) as the following forms,

$$-zy_{t_r} + \tilde{H}_{r,x} + 2zy\tilde{G}_r = 0, \quad (6.10)$$

$$\tilde{F}_{r,x} = 2z\tilde{G}_r, \quad (6.11)$$

$$\tilde{G}_{r,x} = \tilde{H}_r - y\tilde{F}_r, \quad (6.12)$$

and

$$F_{n,x} = 2zG_n, \quad (6.13)$$

$$H_{n,x} = -2zyG_n, \quad (6.14)$$

$$G_{n,x} = H_n - yF_n. \quad (6.15)$$

From (6.13)-(6.15), we may compute

$$\frac{d}{dx} \det(B_n(z)) = -\frac{1}{z} \frac{d}{dx} \left( zG_n(z)^2 - F_n(z)H_n(z) \right) = 0, \quad (6.16)$$

and meanwhile Lemma 6.3 gives

$$\frac{d}{dt_r} \det(B_n(z)) = -\frac{1}{z} \frac{d}{dt_r} \left( zG_n(z)^2 - F_n(z)H_n(z) \right) = 0. \quad (6.17)$$

Hence,  $zG_n(z)^2 - F_n(z)H_n(z)$  is independent of variables both  $x$  and  $t_r$ , which implies

$$zG_n(z)^2 - F_n(z)H_n(z) = k^2(z). \quad (6.18)$$

This reveals that the fundamental identity (3.11) still holds in the time-dependent context. Consequently the hyperelliptic curve  $\mathcal{K}_n$  is still available by (5.2).

Next, we define

$$m_+(P, x, t_r) = \frac{k(P) + \sqrt{z}G_n(z)}{F_n(z)}, \quad \pi(P) = z \quad (6.19)$$

and

$$m_-(P, x, t_r) = \frac{-k(P) + \sqrt{z}G_n(z)}{F_n(z)}, \quad \pi(P) = z. \quad (6.20)$$

The properties of the Weyl  $m$ -functions  $m_{\pm}(P, x, t_r)$  are summarized as follows.

**Lemma 6.1** *The Weyl  $m$ -functions  $m_{\pm}(P, x, t_r)$  satisfy the following Riccati equation,*

$$M_x(P, x, t_r) + z^{1/2}M(P, x, t_r)^2 = -z^{1/2}y(x, t_r). \quad (6.21)$$

Moreover,

$$m_+(P, x, t_r) + m_-(P, x, t_r) = \frac{2\sqrt{z}G_n(z)}{F_n(z)}, \quad (6.22)$$

$$m_+(P, x, t_r) - m_-(P, x, t_r) = \frac{2k(P)}{F_n(z)}, \quad (6.23)$$

$$m_+(P, x, t_r)m_-(P, x, t_r) = \frac{H_n(z)}{F_n(z)}. \quad (6.24)$$

**Proof.** The proof of (6.21) is identical to (5.6). The relations (6.22)-(6.24) are an immediate consequence of the definitions of  $m_{\pm}$  and the fundamental identity (6.18).  $\square$

**Lemma 6.2** *Assume that (6.4) and (6.5) hold. Let  $P \in \mathcal{R} \setminus \{P_{\infty}\}$  and  $(x, t_r) \in \mathbb{R}^2$ . Then the function  $M(P, x, t_r)$  satisfies the following differential equations*

$$M_{t_r}(P, x, t_r) = -z^{-\frac{1}{2}}\tilde{H}_r(z) + 2\tilde{G}_r(z)M(P, x, t_r) - z^{-\frac{1}{2}}\tilde{F}_r(z)M(P, x, t_r)^2, \quad (6.25)$$

and

$$\begin{aligned} M_{t_r}(P, x, t_r) &= (-z^{-\frac{1}{2}}\tilde{G}_r(z) + z^{-1}\tilde{F}_r(z)M(P, x, t_r))_x \\ &= -z^{-\frac{1}{2}}\tilde{H}_r(z) + z^{-\frac{1}{2}}y\tilde{F}_r(z) + z^{-1}(\tilde{F}_r(z)M(P, x, t_r))_x. \end{aligned} \quad (6.26)$$

**Proof.** Using (6.10)-(6.12) and (6.21), by a straightforward but rather lengthy calculation, we infer

$$(\partial_x + 2z^{\frac{1}{2}}M)(M_{t_r} + z^{-\frac{1}{2}}\tilde{H}_r - 2\tilde{G}_rM + z^{-\frac{1}{2}}\tilde{F}_rM^2) = 0. \quad (6.27)$$

Hence

$$M_{t_r} + z^{-\frac{1}{2}}\tilde{H}_r - 2\tilde{G}_rM + z^{-\frac{1}{2}}\tilde{F}_rM^2 = C \exp\left(-2 \int^x z^{\frac{1}{2}}M \, dx'\right), \quad (6.28)$$

where the left-hand side is meromorphic in a neighborhood of  $P_{\infty}$ , while the right-hand side is meromorphic near  $P_{\infty}$  only if  $C = 0$ . This proves (6.25). Next, by using (6.11) and (6.21), we obtain

$$\begin{aligned} z^{-\frac{1}{2}}y\tilde{F}_r + z^{-1}(\tilde{F}_rM)_x &= z^{-\frac{1}{2}}y\tilde{F}_r + z^{-1}\tilde{F}_{r,x}M + z^{-1}\tilde{F}_rM_x \\ &= 2\tilde{G}_rM - z^{-\frac{1}{2}}\tilde{F}_rM^2. \end{aligned} \quad (6.29)$$

Combining this result with (6.25), we conclude that (6.26) holds. Alternatively, more efficiently method to prove (6.25) and (6.26) can be found in our paper [16].  $\square$

Next, we study the time evolution of  $F_n$ ,  $G_n$  and  $H_n$  by using zero-curvature equations (6.10)-(6.12) and (6.13)-(6.15).

**Lemma 6.3** *Assume that (6.4) and (6.5) hold. Then*

$$F_{n,t_r} = 2(G_n\tilde{F}_r - \tilde{G}_rF_n), \quad (6.30)$$

$$zG_{n,t_r} = H_n \tilde{F}_r - \tilde{H}_r F_n, \quad (6.31)$$

$$H_{n,t_r} = 2(H_n \tilde{G}_r - G_n \tilde{H}_r). \quad (6.32)$$

Equations (6.30) – (6.32) imply

$$-B_{n,t_r} + [\tilde{B}_r, B_n] = 0. \quad (6.33)$$

**Proof.** Differentiating both sides of (6.23) with respect to  $t_r$  leads to

$$(m_+ - m_-)_{t_r} = -2k(P)F_{n,t_r}F_n^{-2}. \quad (6.34)$$

On the other hand, by (6.22), (6.23) and (6.25), the left-hand side of (6.34) equals to

$$\begin{aligned} m_{+,t_r} - m_{-,t_r} &= 2\tilde{G}_r(m_+ - m_-) - z^{-\frac{1}{2}}\tilde{F}_r(m_+^2 - m_-^2) \\ &= 4k(P)(\tilde{G}_r F_n - \tilde{F}_r G_n)F_n^{-2}. \end{aligned} \quad (6.35)$$

Combining (6.34) with (6.35) yields (6.30). Similarly, Differentiating both sides of (6.22) with respect to  $t_r$  gives

$$(m_+ + m_-)_{t_r} = 2z^{\frac{1}{2}}(G_{n,t_r}F_n - G_n F_{n,t_r})F_n^{-2}, \quad (6.36)$$

Meanwhile, by (6.22), (6.23) and (6.25), the left-hand side of (6.36) equals to

$$\begin{aligned} m_{+,t_r} + m_{-,t_r} &= 2\tilde{G}_r(m_+ + m_-) - z^{-\frac{1}{2}}\tilde{F}_r(m_+^2 + m_-^2) - 2z^{-\frac{1}{2}}\tilde{H}_r \\ &= -2z^{\frac{1}{2}}G_n F_n^{-2}F_{n,t_r} + 2z^{-\frac{1}{2}}F_n^{-1}(-\tilde{H}_r F_n + \tilde{F}_r H_n) \end{aligned} \quad (6.37)$$

Thus, (6.31) clearly follows by (6.36) and (6.37). Next, differentiating both sides of (6.24) with respect to  $t_r$  yields

$$(m_+ m_-)_{t_r} = (H_{n,t_r}F_n - H_n F_{n,t_r})F_n^{-2}. \quad (6.38)$$

By using (6.22), (6.24) and (6.25), we compute the left-hand side of (6.38), obtaining

$$\begin{aligned} (m_+ m_-)_{t_r} &= -z^{-\frac{1}{2}}\tilde{H}_r(m_+ + m_-) + 4\tilde{G}_r m_+ m_- \\ &\quad - z^{-\frac{1}{2}}\tilde{F}_r m_+ m_- (m_+ + m_-) \\ &= 2(\tilde{G}_r H_n - \tilde{H}_r G_n)F_n^{-1} - H_n F_{n,t_r}F_n^{-2}, \end{aligned} \quad (6.39)$$

and hence (6.32) holds. Finally, a direct calculation shows that (6.30)-(6.32) are equivalent to (6.33).  $\square$

The properties of the  $x$ -motion and  $t_r$ -motion of the poles  $P_j(x, t_r)$  now reads as follows.

**Theorem 6.4** Assume that (6.4) and (6.5) hold. Then, for every  $j = 1, \dots, n$ ,

$$P_{j,x}(x, t_r) = \frac{2(-1)^n k(P_j(x, t_r)) \sqrt{P_j(x, t_r)} \prod_{i=1}^n P_i(x, t_r)}{k(0) \prod_{j \neq i} (P_j(x, t_r) - P_i(x, t_r))}, \quad (6.40)$$

and

$$\begin{aligned} P_{j,t_r}(x, t_r) &= \frac{2(-1)^n k(P_j(x, t_r)) \tilde{F}_r(P_j(x, t_r)) \prod_{i=1}^n P_i(x, t_r)}{k(0) \sqrt{P_j(x, t_r)} \prod_{j \neq i} (P_j(x, t_r) - P_i(x, t_r))} \\ &= \frac{\tilde{F}_r(P_j(x, t_r))}{P_j(x, t_r)} P_{j,x}(x, t_r). \end{aligned} \quad (6.41)$$

**Proof.** It suffices to focus on (6.41), since the proof procedure for (6.40) is analogous to (5.7). Differentiating on both sides of (5.12) with respect to  $t_r$  yields

$$F_{n,t_r}(P_j) = -\frac{P_{j,t_r}}{\sqrt{y}} \prod_{i \neq j} (P_j - P_i). \quad (6.42)$$

On the other hand, considering (6.18), we compute (6.30) at  $P_j$ ,

$$F_{n,t_r}(P_j) = 2\tilde{F}_r(P_j)G_n(P_j) = 2\tilde{F}_r(P_j)\frac{k(P_j)}{\sqrt{P_j}}. \quad (6.43)$$

Hence, combining (6.42) and (6.43) leads to (6.41).  $\square$

**Remark 6.5** A closer look at Theorem 6.4 reveals that the pole motion (both for the  $x$ -motion and the  $t_r$ -motion) can be determined by solving only first order differential equations

$$\frac{\partial P_j}{\partial x} = U_1(P_j) \quad \text{and} \quad \frac{\partial P_j}{\partial t_r} = U_2(P_j),$$

where  $U_1$  and  $U_2$  are bounded continuous functions defined on  $\mathbb{R}$ .

Now we shall provide the algebro-geometric formula for time-dependent HS solutions  $u(x, t_r)$ .

**Theorem 6.6** Assume that (6.4) and (6.5) hold. Then the  $r$ -th order HS equation (4.11) admits a global solution  $u(x, t_r)$  of algebro-geometric type, when the initial condition  $u(x, t_0) = u_0(x)$  is given by the solution of the stationary HS equation of order  $n$ . The function  $y_0(x) = y(x, t_0) = u_{xx}(x, t_0)/2$

lies in the isospectral class given by Hypotheses 2.4 for the Sturm-Liouville operator  $L\psi_1 = \psi_{1,xx} = -zy\psi_1$ .

In particular,

$$u(x, t_r) = \sum_{j=1}^n \frac{1}{P_j(x, t_r)} - \sum_{m=1}^{2n} \frac{1}{z_m}, \quad (6.44)$$

where the pole motion is completely determined from (6.40) and (6.41).

Moreover, for every  $t_r \in \mathbb{R}$ , the function  $y_{t_r}(x) = y(x, t_r) = u_{xx}(x, t_r)/2$  lies in the same isospectral class as  $y_0(x)$ . That is, time evolution of the solutions of the  $r$ -th order HS equation define densities lying in the same isospectral class, and this isospectral class depends only on the initial data.

**Proof.** The proof of (6.44) is analogous to Theorem 5.5. To show that all the solutions define densities lying in the same isospectral class, it is sufficient to observe that the  $t_r$ -evolution of the poles  $P_j(x, t_r)$  implies that, if we start from a time  $\tilde{t}_r$ , then the motion  $t_r \mapsto P_j(x, t_r)$  remains in the resolvent interval  $[z_{2j-1}, z_{2j}]$  ( $j = 1, \dots, n$ ). Hence, we can apply Theorem 2.6 to conclude the result.  $\square$

## 7 Jacobian flows and pole motion on the generalized Jacobian

In this section we move our attention to the generalized Jacobian variety  $J_0(\mathcal{R})$  of the Riemann surface  $\mathcal{R}$ . The aim is to make clearer the structure of the solutions  $u(x, t_r)$  given in (6.44).

We first give a short description of what a generalized Jacobian is, for more details, one can refer to [17, 34].

Let  $\mathcal{R}$  denote a hyperelliptic Riemann surface of genus  $n$  with a standard homology basis  $a_i, b_i$  ( $i = 1, \dots, n$ ), and  $dw_1, \dots, dw_n$  be a basis of normalized holomorphic differentials on  $\mathcal{R}$ . The normalized differentials  $dw_i$ , means

$$\int_{a_i} dw_j = \delta_{i,j}.$$

We construct a singular Riemann surface  $\mathcal{R}_0$ , by pinching a nonzero homology cycle at a ramification point  $Q_0$  on  $\mathcal{R}$ . Roughly speaking, a standard basis of holomorphic differentials is no longer sufficient to describe the structure of the Jacobian variety connecting to this new surface  $\mathcal{R}_0$ : such a

Jacobian variety is called generalized Jacobian variety, and we denote it by  $J_0(\mathcal{R})$ .

To solve inverse problems on generalized Jacobian variety, we need the normalized differential of the second kind  $dw_{Q_0}^{(2)}$  on  $\mathcal{R}$  having a double pole at  $Q_0$ , with principal part  $\lambda_{Q_0}^{-2}d\lambda_{Q_0}$ . The normalization means

$$\int_{a_i} dw_{Q_0}^{(2)} = 0, \quad (i = 1, \dots, n).$$

Let us give a more precise definition of  $J_0(\mathcal{R})$ . We take the set  $Symm^{n+1}(\mathcal{R})$  of unordered  $n+1$ -tuples of points on  $\mathcal{R}$ , such tuples are called divisors. Two divisors  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are identified if  $\mathcal{D}_1 - \mathcal{D}_2$  is the divisor of a meromorphic function on  $\mathcal{R}_0$ . For a more detailed treatment we refer to [38]. The map  $\tilde{I}_0$

$$\left( \sum_{i=0}^n \int_{P^*}^{P_i} (dw_{Q_0}^{(2)}, dw_1, \dots, dw_n) \right) \in \mathbb{C}^{n+1}$$

sends each  $(P_0, P_1, \dots, P_n) \in Symm^{n+1}(\mathcal{R})$  to a complex vector  $(z_0, s) \in \mathbb{C} \times \mathbb{C}^n$ , where  $P^*$  is a fixed initial point different from the ramification points of  $\mathcal{R}$ .

Let  $\Lambda_0 \subset \mathbb{C}^{n+1}$  be the  $\mathbb{Z}$ -lattice spanned by all vectors of the form

$$\int_{\alpha} (dw_{Q_0}^{(2)}, dw_1, \dots, dw_n), \quad \alpha = a_0, a_1, \dots, a_n, b_1, \dots, b_n.$$

Clearly  $\int_{a_0} dw_{Q_0}^{(2)} = 0$  and  $\int_{a_0} dw_i = 0$  ( $i = 1, \dots, n$ ), where  $a_0$  is a sufficiently small simple closed curve centered at  $Q_0 \in \mathcal{R}$ <sup>1</sup>. It turns out that the rank of  $\Lambda_0$  is  $2n$ . The generalized Jacobian  $J_0(\mathcal{R})$  is defined by,  $J_0(\mathcal{R}) = \mathbb{C}^{n+1}/\Lambda_0$ . It can be shown that  $J_0(\mathcal{R}) \cong \mathbb{C}^* \times \mathbb{C}^n$ . For more details about generalized Jacobian, one can see [17, 38].

Given this construction, we can solve the Jacobi inversion problem when in presence of nonholomorphic differentials. We define the generalized Riemann Theta function

$$\Theta_0(\underline{z}) = z_0 \Theta(s) + \partial_{\omega_n} \Theta(s), \quad \underline{z} = (z_0, s) \in \mathbb{C} \times \mathbb{C}^n,$$

where  $\Theta(s)$  is the classical Riemann Theta function associated to the Riemann surface  $\mathcal{R}$  of genus  $n$ , and  $\omega_n$  is defined in (8.5). From the standard

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<sup>1</sup> $a_0$  is required to bound a disc in  $\mathcal{R}$  centered at  $Q_0$ , and the support of  $a_0$  is required to be disjoint from the supports of  $a_i$  and  $b_i$  ( $i = 1, \dots, n$ ).



theory, we know that every symmetric function of divisors  $(P_0, P_1, \dots, P_n)$  can be expressed in terms of a Theta quotient.

We define the generalized Abel map,  $I_0 : \text{Symm}^{n+1}(\mathcal{R}) \rightarrow J_0(\mathcal{R}) = \mathbb{C}^{n+1}/\Lambda_0$ , by

$$I_0(P_0, P_1, \dots, P_n) = \left( \sum_{i=0}^n \int_{P^*}^{P_i} (dw_{Q_0}^{(2)}, dw_1, \dots, dw_n) \right) \mod \Lambda_0.$$

Then it is clear that by setting  $P_0 = P^*$ , the generalized Abel map  $I_0$  defines an isomorphism between the space  $\text{Symm}^n(\mathcal{R})$  and a non compact subvariety of  $J_0(\mathcal{R})$ , given by the locus of the zeros of the generalized Theta function  $\Theta_0$ . That is,  $I_0$  sends divisors of degree  $n$  into vectors in  $J_0(\mathcal{R})$  of the form

$$I_0(P_1, \dots, P_n) = \tilde{z} = \left( \sum_{i=1}^n \int_{P^*}^{P_i} (dw_{Q_0}^{(2)}, dw_1, \dots, dw_n) \right) \in \mathbb{C}^{n+1}$$

such that  $\Theta_0(\tilde{z} - \Delta_0) = 0$ , where  $\Delta_0$  is the generalized vector of Riemann constants.

The subvariety  $I_0(\text{Symm}^n(\mathcal{R})) \subset J_0(\mathcal{R})$  is denoted by  $\Upsilon_0$ . It turns out that every symmetric rational function of divisors of degree  $n$  can be expressed as the restriction of the corresponding function of divisors of degree  $n+1$  by setting  $P_0 = P^*$ , hence it can be written as a theta quotient. We will study these facts in more detail in Section 8.

Now we turn to our case,  $\mathcal{R}$  is the Riemann surface described by the following algebraic relation,

$$w^2(z) - z \prod_{m=1}^{2n} (z - z_m) = 0,$$

$Q_0 = 0$ , and the corresponding nonholomorphic differential is  $dw_0 = dw_{Q_0}^{(2)}$ . Let  $c_j = \pi^{-1}[z_{2j-1}, z_{2j}]$  ( $j = 1, \dots, n$ ), each  $c_j$  is a simple closed curve in  $\mathcal{R}$ . The product  $c_1 \times c_2 \times \dots \times c_n$  is a real  $n$ -torus, and embeds into  $\text{Symm}^n(\mathcal{R})$ , and hence into  $\Upsilon_0$  through the restricted Abel map. Next, we choose  $P_1 \in c_1, \dots, P_n \in c_n$ , then the correspondence  $(P_1, \dots, P_n) \mapsto \sum_{i=1}^n \int_{P^*}^{P_i} (dw_0, \dots, dw_{n-1})$  maps  $c_1 \times \dots \times c_n$  into a curvilinear parallelogram  $\ell \subset \mathbb{C}^n$ , hence the remaining coordinate  $\sum_{i=1}^n \int_{P^*}^{P_i} dw_n$  can be viewed as

a nonlinear transcendental function on  $\ell$  of  $\sum_{i=1}^n \int_{P^*}^{P_i} (dw_0, dw_1, \dots, dw_{n-1})$ ,  
such that  $\sum_{i=1}^n \int_{P^*}^{P_i} (dw_0, dw_1, \dots, dw_{n-1}, dw_n) \in \Upsilon_0$ .

Next, we will study the nature of the motion of the poles  $P_j(x, t_r)$  on  $J_0(\mathcal{R})$ . To this aim, let  $\mathcal{P} = (P_1, \dots, P_n) \in \mathbb{C}^n$  be an arbitrary  $n$ -tuple of distinct points. We introduce the following symmetric functions:

$$\varsigma_i(\mathcal{P}) = (-1)^i \sum_{\ell \in \Lambda_i} P_{\ell_1} \dots P_{\ell_i}, \quad \ell = (\ell_1, \dots, \ell_i), \quad 1 \leq i \leq n, \quad (7.1)$$

where  $\Lambda_i = \{\ell \in \mathbb{N}^i \mid 1 \leq \ell_1 < \dots < \ell_i \leq n\}$ ;

$$\sigma_i^{(j)}(\mathcal{P}) = (-1)^i \sum_{\ell \in \Lambda_i^{(j)}} P_{\ell_1} \dots P_{\ell_i}, \quad \ell = (\ell_1, \dots, \ell_i), \quad 1 \leq i \leq n-1, \quad (7.2)$$

where  $\Lambda_i^{(j)} = \{\ell \in \mathbb{N}^i \mid 1 \leq \ell_1 < \dots < \ell_i \leq n, \ell_k \neq j\}$ .

The general form of Lagrange's interpolation theorem then reads as follows.

**Theorem 7.1 (Lagrange interpolation formula)** *Let  $\mathcal{P} = (P_1, \dots, P_n) \in \mathbb{C}^n$  be a  $n$ -tuple of distinct points. Then for every  $k = 1, \dots, n+1$ ,  $i = 0, \dots, n-1$  and  $j = 1, \dots, n$ , we have*

$$\sum_{j=1}^n \frac{P_j^{k-1}}{\prod_{s \neq j} (P_j - P_s)} \sigma_i^{(j)}(\mathcal{P}) = \delta_{k,n-i} - \varsigma_{i+1}(\mathcal{P}) \delta_{k,n+1}. \quad (7.3)$$

The simplest Lagrange interpolation formula reads in the case  $i = 0$ ,

$$\sum_{j=1}^n \frac{P_j^{k-1}}{\prod_{s \neq j} (P_j - P_s)} = \delta_{k,n}, \quad k = 1, \dots, n.$$

For use in present paper, we recall some important properties of the functions  $\sigma_i^{(j)}$  and  $\varsigma_i$ . These results can be found in [12], here we omit the proofs.

**Lemma 7.2** *Let  $\mathcal{P} = (P_1, \dots, P_n) \in \mathbb{C}^n$  be a  $n$ -tuple of distinct points.*

Then

$$\begin{aligned}
(i) \quad & \varsigma_{i+1}(\mathcal{P}) + P_j \sigma_i^{(j)}(\mathcal{P}) = \sigma_{i+1}^{(j)}(\mathcal{P}), \quad i = 0, \dots, n-1, \quad j = 1, \dots, n. \\
(ii) \quad & \sum_{i=0}^r \varsigma_{r-i}(\mathcal{P}) P_j^i = \sigma_r^{(j)}(\mathcal{P}), \quad r = 0, \dots, n, \quad j = 1, \dots, n. \\
(iii) \quad & \sum_{i=0}^{r-1} \sigma_{r-1-i}^{(j)}(\mathcal{P}) z^i = \frac{1}{z - P_j} \left( \sum_{i=0}^r \varsigma_{r-i}(\mathcal{P}) z^i - \sigma_r^{(j)}(\mathcal{P}) \right).
\end{aligned}$$

We define the (non normalized) differentials

$$d\omega_{s-1} = \frac{z^{s-2}}{2\sqrt{z} k(z)} dz, \quad s = 1, \dots, n.$$

Note that  $d\omega_0$  is a differential of the second kind, having a double pole at 0, with coefficient  $(1/\sqrt{\prod z_m})$  of corresponding principal part.

Moreover, let

$$\omega_{s-1}(x, t_r) = \sum_{i=1}^n \int_{P^*}^{P_i(x, t_r)} d\omega_{s-1}, \quad s = 1, \dots, n.$$

Then differentiating each  $\omega_{s-1}$  with respect to  $x$ , we have

$$\frac{\partial \omega_{s-1}(x, t_r)}{\partial x} = \sum_{j=1}^n \frac{P_j^{s-2}(x, t_r)}{2\sqrt{P_j(x, t_r)} k(P_j(x, t_r))} \frac{\partial P_j(x, t_r)}{\partial x}. \quad (7.4)$$

Substituting (6.40) into (7.4) and taking account into Theorem 7.1 yields

$$\frac{\partial \omega_{s-1}(x, t_r)}{\partial x} = \frac{-1}{k(0)} \sum_{j=1}^n \frac{P_j^{s-1}(x, t_r) \sigma_{n-1}^{(j)}(\mathcal{P})}{\prod_{i \neq j} (P_j(x, t_r) - P_i(x, t_r))} = \frac{-1}{k(0)} \delta_{s,1}, \quad (7.5)$$

which implies the following relation:

$$\frac{\partial \omega_{s-1}(x, t_r)}{\partial x} = \begin{cases} -\frac{1}{k(0)}, & s = 1, \\ 0, & s = 2, \dots, n. \end{cases} \quad (7.6)$$

Hence we obtain

$$\omega_{s-1}(x, t_r) = \begin{cases} c_0(t_r) - \frac{1}{k(0)} x, & s = 1, \\ c_{s-1}(t_r), & s = 2, \dots, n. \end{cases} \quad (7.7)$$

This result shows that the  $x$ -motion is constant with respect to  $n - 1$  coordinates, while it is linear with respect to  $x$  on the remaining coordinate, which corresponds to the differential of the second kind.

Now we investigate the  $t_r$ -motion of the poles  $P_j(x, t_r)$ . Recall (5.12):

$$F_n(z) = \frac{1}{\sqrt{y(x, t_r)}} \prod_{i=1}^n (z - \pi(P_i(x, t_r))).$$

By the construction of  $\tilde{F}_r(z)$ , we know that for every  $r = 1, \dots, n$ , the polynomial  $\tilde{F}_r(z)$  can be obtained by truncating the polynomial  $F_n(z)$  at the degree  $r$ , that is

$$\tilde{F}_r(z) = F_n(z) - z^{r+1} \left[ \frac{F_n(z)}{z^{r+1}} \right]_p, \quad (7.8)$$

where  $[ ]_p$  denotes the polynomial part. Hence, we conclude that

$$\tilde{F}_r(P_j(x, t_r)) = -P_j^{r+1}(x, t_r) \left[ \frac{F_n(P_j(x, t_r))}{z^{r+1}} \right]_p = -\frac{P_j^{r+1}(x, t_r)}{\sqrt{y(x, t_r)}} \sigma_{n-r-1}^{(j)}(\mathcal{P}). \quad (7.9)$$

Inserting (7.9) into (6.41) yields

$$\begin{aligned} P_{j,t_r}(x, t_r) &= -\frac{P_j^r(x, t_r)}{\sqrt{y(x, t_r)}} \sigma_{n-r-1}^{(j)}(\mathcal{P}) P_{j,x}(x, t_r) \\ &= \frac{P_j^r(x, t_r) \sigma_{n-r-1}^{(j)}(\mathcal{P}) k(P_j(x, t_r)) \sqrt{P_j(x, t_r)}}{\prod_{i \neq j} (P_j(x, t_r) - P_i(x, t_r))}. \end{aligned} \quad (7.10)$$

As before, differentiating each  $\omega_{s-1}$  with respect to  $t_r$  gives

$$\frac{\partial \omega_{s-1}(x, t_r)}{\partial t_r} = \sum_{j=1}^n \frac{P_j^{s-2}(x, t_r)}{2\sqrt{P_j(x, t_r)} k(P_j(x, t_r))} \frac{\partial P_j(x, t_r)}{\partial t_r}, \quad (7.11)$$

and insertion of (7.10) into (7.11) yields the beautiful relation

$$\frac{\partial \omega_{s-1}(x, t_r)}{\partial t_r} = \sum_{j=1}^n \frac{P_j^{s+r-2}(x, t_r) \sigma_{n-r-1}^{(j)}(\mathcal{P})}{\prod_{i \neq j} (P_j(x, t_r) - P_i(x, t_r))}, \quad (7.12)$$

where  $s = 1, \dots, n$  and  $r = 1, \dots, n - 1$ .

After a short computation, we arrive at

$$d\omega_{s-1} = \delta_{s-1,1} dt_r - \sum_{l=0}^{r-1} \left( \sum_{j=1}^n \frac{P_j^{s+l-2} \varsigma_{n-l-1}(\mathcal{P})}{\prod_{i \neq j} (P_j - P_i)} \right) dt_r. \quad (7.13)$$

Next, we introduce the notation  $\alpha_r = n - r + 1$ . Then (7.13) gives:  
For every  $s = 0, \dots, \alpha_r - 1$ ,

$$d\omega_s = \delta_{s,1} dt_r.$$

For  $k = 0, \dots, r - 2$ ,

$$d\omega_{\alpha_r+k} = - \sum_{h=n-r}^{n-1} \left( \sum_{j=1}^n \frac{P_j^{k+h} \varsigma_{2n-r-h-1}(\mathcal{P})}{\prod_{i \neq j} (P_j - P_i)} \right) dt_r.$$

If we set

$$dt_{r,h} = \varsigma_{2n-r-h-1}(\mathcal{P}) dt_r$$

and

$$\mathcal{H}_{h+k}(\mathcal{P}) = \sum_{j=1}^n \frac{P_j^{k+h}}{\prod_{i \neq j} (P_j - P_i)},$$

then we obtain

$$d\omega_{\alpha_r+k} = - \sum_{h=n-r}^{n-1} \mathcal{H}_{h+k}(\mathcal{P}) dt_{r,h}.$$

As long as  $k + h$  does not reach the value  $n - 1$ , then  $\mathcal{H}_{h+k}(\mathcal{P}) = 0$ . Hence, for every fixed  $r$ , we have the following triangular structure for the pole motion:

$$d\omega_s(x, t_r) = \begin{cases} -\frac{1}{k(0)} dx, & s = 0, \\ dt_r, & s = 1, \\ 0, & s = 2, \dots, \alpha_r - 1, \\ -\sum_{h=n-r}^{n-1} \mathcal{H}_{h+k}(\mathcal{P}) dt_{r,h}, & s = \alpha_r + k, \quad k = 0, \dots, r - 2. \end{cases} \quad (7.14)$$

A closer look at (7.14) implies the main differences between our formulas for the pole motion in the generalized Jacobian and those in [7]:

1. In our context, time motion is confined to the holomorphic coordinates, i.e.  $\omega_1(x, t_r) = \omega_1(t_r), \dots, \omega_{n-1}(x, t_r) = \omega_{n-1}(t_r)$ , while the  $x$ -motion evolves only in the meromorphic one, namely  $\omega_0(x, t_r) = \omega_0(x)$ ; this shows a complete separation between spatial and time motions.
2. For every  $r = 1, \dots, n-1$ , the motion is linear with respect to  $x$ , with no need of any linearizing change of variables.
3. The classical one dimensional HS equation corresponds to the case  $r = 1$ , then the motion on the generalized Jacobian is remarkably simple:

$$\omega_s(x, t_1) = \begin{cases} -\frac{1}{k(0)}x + \chi_0, & s = 0, \\ t_1 + \chi_1, & s = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\chi_0$  and  $\chi_1$  are constant phases.

## 8 Theta function representation for the solution $u(x, t_r)$ on the generalized Jacobian

In our final section we present expressions for all the elementary symmetric functions of  $n$  distinct points on  $\mathcal{R}$  in terms of Riemann theta function. In particular, we obtain the theta function representation for the solution  $u(x, t_r)$  of the HS hierarchy.

Let

$$d\omega_{k-1} = \frac{z^{k-2}}{2\sqrt{z} k(z)} dz, \quad k = 1, \dots, n. \quad (8.1)$$

We note that for  $k > 1$ ,  $d\omega_{k-1}$  is a non normalized holomorphic differential on  $\mathcal{R}$ . While  $k = 1$ ,  $d\omega_0$  is a differential of the second kind, having a double pole at 0 with principal part  $(1/\sqrt{\prod z_m})\lambda_0^{-2}d\lambda_0$ , in terms of the local coordinate  $\lambda_0 = z^{1/2}$  near 0.

Moreover, we denote by  $dw_1, \dots, dw_g$  a normalized basis of holomorphic differentials on  $\mathcal{R}$ , and by  $dw_0$  the normalized differential of the second kind having a double pole at 0 with principal part  $\lambda_0^{-2}d\lambda_0$ , in terms of the local coordinate  $\lambda_0 = z^{1/2}$  near 0.

Let  $\mathcal{P} = (P_1, \dots, P_n)$ , where  $P_i$  ( $i = 1, \dots, n$ ) are distinct points on  $\mathcal{R}$ . For our convenience, we write  $P_i$  instead of  $\pi(P_i)$  if no confusion arises.

Define the variables  $\alpha_1, \dots, \alpha_n$ , such that

$$\frac{\partial P_j}{\partial \alpha_k} = \frac{2P_j^{3/2} \sigma_{n-k}^{(j)}(\mathcal{P}) k(P_j)}{\prod_{i \neq j} (P_j - P_i)}, \quad j, k = 1, \dots, n, \quad (8.2)$$

where  $P_j = P_j(\alpha_1, \dots, \alpha_n)$ . Next, we give the explicit formulas for  $\alpha_1, \dots, \alpha_n$ . Combining (8.1) and (8.2), we obtain

$$\begin{aligned} \frac{\partial}{\partial \alpha_k}(\omega_{k-1}) &= \frac{\partial}{\partial \alpha_k} \left( \sum_{j=1}^n \int_{P^*}^{P_j(\alpha_1, \dots, \alpha_n)} \frac{z^{k-2}}{2\sqrt{z} k(z)} dz \right) \\ &= \sum_{j=1}^n \frac{P_j^{k-2}}{2\sqrt{P_j} k(P_j)} \frac{\partial P_j}{\partial \alpha_k} = \sum_{j=1}^n \frac{P_j^{k-1} \sigma_{n-k}^{(j)}(\mathcal{P})}{\prod_{i \neq j} (P_j - P_i)} \\ &= \delta_{k,k} - \varsigma_{n-k+1} \delta_{k,n+1} = 1. \end{aligned} \quad (8.3)$$

Hence, we have

$$\alpha_k - \alpha_k^0 = \sum_{j=1}^n \int_{P^*}^{P_j} d\omega_{k-1} = \omega_{k-1}, \quad k = 1, \dots, n. \quad (8.4)$$

Next, we introduce the additional function

$$\alpha_{n+1} = \sum_{j=1}^n \int_{P^*}^{P_j(\alpha_1, \dots, \alpha_n)} \frac{z^{n-1}}{2\sqrt{z} k(z)} dz = \omega_n(\mathcal{P}). \quad (8.5)$$

Differentiating (8.5) with respect to  $\alpha_k$  on both sides gives

$$\frac{\partial \alpha_{n+1}}{\partial \alpha_k} = \sum_{j=1}^n \frac{P_j^n}{\prod_{i \neq j} (P_j - P_i)} \sigma_{n-k}^{(j)}(\mathcal{P}) = \delta_{n+1,k} - \varsigma_{n-k+1}(\mathcal{P}) = -\varsigma_{n-k+1}(\mathcal{P}). \quad (8.6)$$

Equation (8.6) shows that all the symmetric functions of the poles  $P_1, \dots, P_n$ , which are restrictions of symmetric functions of  $n+1$  points in  $J_0(\mathcal{R})$ , can be determined by differentiating the  $n$ -th non normalized Abel coordinate with respect to the  $(k-1)$ -th one.

There is a normalizing matrix  $D \in Sl(n, \mathbb{C})$  such that

$$\underline{\omega} = D \underline{w}, \quad (8.7)$$

where  $\underline{\omega}$  and  $\underline{w}$  denote the column vectors of the Abel coordinates  $(\omega_1, \dots, \omega_n)$  and  $(w_1, \dots, w_n)$  respectively. Moreover, there are constants  $\eta_1, \dots, \eta_n \in \mathbb{C}$ , such that

$$d\omega_0 = k(0)d\omega_0 + \sum_{i=1}^n \eta_i dw_i, \quad (8.8)$$

where

$$\eta_i = -k(0) \int_{a_i} d\omega_0.$$

For  $s, k = 1, \dots, n$ , a direct calculation yields

$$\frac{\partial \alpha_s}{\partial \alpha_k} = \sum_{j=1}^n \frac{P_j^{s-1} \sigma_{n-k}^{(j)}(\mathcal{P})}{\prod_{i \neq j} (P_j - P_i)} = \delta_{s,k}. \quad (8.9)$$

We denote by  $D = (\gamma_{rs})$  and  $D^{-1} = (\beta_{rs})$ . Then from (8.9) we infer

$$\frac{\partial w_s}{\partial \alpha_k} = \sum_{j=1}^n \beta_{sj} \frac{\partial w_j}{\partial \alpha_k} = \sum_{j=1}^{n-1} \beta_{sj} \frac{\partial \alpha_{j+1}}{\partial \alpha_k} + \beta_{sn} \frac{\partial \alpha_{n+1}}{\partial \alpha_k} = \beta_{s,k-1} - \beta_{sn} \varsigma_{n-k+1}(\mathcal{P}), \quad (8.10)$$

where  $\beta_{s0} = 0$ .

Now we intend to determine the dependence of  $\frac{\partial \omega_n}{\partial \alpha_k}$  with respect to the classical Riemann theta function  $\Theta(s)$ , where  $s = s(\mathcal{P}) = I(P_1, \dots, P_n)$  and  $I : \text{Symm}^n(\mathcal{R}) \rightarrow J(\mathcal{R})$  denotes the standard Abel map. Since  $w_0(P_1, \dots, P_n) = -\partial_{\omega_n} \ln \Theta(s(\mathcal{P}))$  (see [7, 9]), then combining (8.7) and (8.8) yields

$$\omega_n = \sum_{s=1}^n \gamma_{ns} w_s = \sum_{s=1}^{n-1} \gamma_{ns} w_s + \frac{\gamma_{nn}}{\eta_n} \left( -\partial_{\omega_n} \ln \Theta(s(\mathcal{P})) - k(0) \alpha_1 - \sum_{i=1}^{n-1} \eta_i w_i \right). \quad (8.11)$$

Then differentiating (8.11) with respect to  $\alpha_k$ , after some computations, we arrive at

$$\left( \sum_{s=1}^n \beta_{sn} \eta_s \right) \varsigma_{n-k+1}(\mathcal{P}) = \left( \sum_{s=1}^n \beta_{s,k-1} \eta_s \right) + k(0) \delta_{1,k} + \frac{\partial^2}{\partial \alpha_k \partial \omega_n} \ln \Theta(s(\mathcal{P})). \quad (8.12)$$

Introducing the notation

$$\xi_k = \left( \sum_{s=1}^n \beta_{sk} \eta_s \right), \quad k = 1, \dots, n.$$

Next, we intend to make clear the meaning of the constants  $\xi_k$ . For this purpose, consider the  $(n+1) \times (n+1)$  matrix

$$D_0 = \begin{pmatrix} 1/k(0) & -\eta_1/k(0) & \dots & -\eta_n/k(0) \\ 0 & & & \\ \vdots & & D & \\ 0 & & & \end{pmatrix}.$$



It is clear that  $D_0$  is a normalizing matrix in the sense that

$$\begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = D_0 \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

By a short computation, one can infer that the first row of the inverse matrix  $D_0^{-1}$  is the vector  $(k(0), \xi_1, \dots, \xi_n)$ . This implies that the constants  $\xi_k$  are those complex numbers such that

$$dw_0 = k(0)d\omega_0 + \sum_{k=1}^n \xi_k d\omega_k. \quad (8.13)$$

Based on the above analysis, we have the following result.

**Lemma 8.1** *Let  $\mathcal{P} = (P_1, \dots, P_n)$  be a  $n$ -tuple of distinct points of  $\mathcal{R}$ . Then for every  $k = 1, \dots, n$ ,*

$$\varsigma_{n-k+1}(\mathcal{P}) = \frac{1}{\xi_n} \left( \xi_{k-1} + k(0)\delta_{1,k} + \frac{\partial^2}{\partial \omega_{k-1} \omega_n} \ln \Theta(s(\mathcal{P})) \right), \quad (8.14)$$

where  $s(\mathcal{P}) = I(P_1, \dots, P_n)$ ,  $I : \text{Symm}^n(\mathcal{R}) \rightarrow J(\mathcal{R})$  is the standard Abel map, and the constants  $\xi_k$  satisfy (8.13), with  $\xi_0 = 0$ ,  $\alpha_k = \omega_{k-1}$ .

For every  $n$ -tuple  $\mathcal{P} = (P_1, \dots, P_n)$  of distinct points of  $\mathcal{R}$ , the symmetric rational function  $\varrho : \text{Symm}^n(\mathcal{R}) \rightarrow \mathbb{C} : \mathcal{P} \mapsto \prod_{i=1}^n \pi(P_i)$  is the restriction to  $\text{Symm}^n(\mathcal{R})$ , obtained by taking  $P_0 = P^*$ , of a symmetric rational function  $\rho : \text{Symm}^{n+1}(\mathcal{R}) \rightarrow \mathbb{C}$ . Following [37], every symmetric rational function  $\rho$  on  $\text{Symm}^{n+1}(\mathcal{R})$  defines a meromorphic function on  $J_0(\mathcal{R})$ . Hence, from (8.14), after some manipulations, we obtain the formula for the symmetric function

$$\varsigma_n(\mathcal{P}) = \prod_{i=1}^n P_i = \gamma \frac{\Theta_0^2 \left( I_0(P_1, \dots, P_n) - \int_{P^*}^0 (w_0, w_1, \dots, w_n) - \Delta_0 \right)}{\Theta_0^4 \left( I_0(P_1, \dots, P_n) - \int_{P^*}^\infty (w_0, w_1, \dots, w_n) - \Delta_0 \right)}, \quad (8.15)$$

where  $\gamma$  is a constant depending only on the choice of the base point  $P^*$  and the genus  $n$  of  $\mathcal{R}$ , and  $\Delta_0$  is the generalized vector of Riemann constants. For notational simplicity, we denote the right-hand side of (8.15) by  $(-1)^n \hat{\Theta}_0(\mathcal{P})$ .

We note that (6.44) can be rewritten as

$$u(x, t_r) = -\frac{\varsigma_{n-1}(\mathcal{P}(x, t_r))}{\varsigma_n(\mathcal{P}(x, t_r))} - \sum_{i=1}^{2n} \frac{1}{z_i}, \quad (8.16)$$

where  $\mathcal{P}(x, t_r) = (P_1(x, t_r), \dots, P_n(x, t_r))$ .

Our main result, the theta function representation of the algebro-geometric solution  $u(x, t_r)$  for the HS hierarchy now follows from the material prepared above.

**Theorem 8.2** *The solution  $u(x, t_r) = u(P_1(x, t_r), \dots, P_n(x, t_r))$  of the  $r$ -th order HS equation, can be written as the following form*

$$u(x, t_r) = \frac{(-1)^{n+1}}{\xi_n \hat{\Theta}_0(\mathcal{P}(x, t_r))} \left( \frac{\partial^2}{\partial \omega_1 \omega_n} \ln \Theta(s(\mathcal{P}(x, t_r))) + \xi_1 \right) - \sum_{i=1}^{2n} \frac{1}{z_i}, \quad (8.17)$$

where  $\mathcal{P}(x, t_r) = (P_1(x, t_r), \dots, P_n(x, t_r))$ ,  $s(\mathcal{P}) = I(P_1(x, t_r), \dots, P_n(x, t_r))$ ,  $\xi_1$  and  $\xi_n$  satisfy (8.13), and  $I : \text{Symm}^n(\mathcal{R}) \rightarrow J(\mathcal{R})$  denotes the standard Abel map.

**Proof.** The expression (8.17) is an immediate consequence of Lemma 8.1, (8.15) and (8.16).  $\square$

**Remark 8.3** *Theorem 8.2 shows that the expression (8.17) depends on the theta quotient (8.15) and the partial derivative with respect to the coordinate  $\omega_1$  of the logarithm of  $w_0(P_1(x, t_r), \dots, P_n(x, t_r))$ . Hence,  $u(x, t_r)$  can be viewed as the restriction of a function, defined on  $J_0(\mathcal{R})$ , to the subvariety  $\Upsilon_0$  given by the locus of the zeros of the generalized theta function  $\Theta_0(z)$ .*

## 9 Conclusions

In this paper, we obtained global solutions of algebro-geometric type for all the equations of a new commuting hierarchy containing the Hunter-Saxton equation. As a main tool we used theta function expressions for all the symmetric functions of points  $P_1, \dots, P_n \in \mathcal{R}$ . Some of these expressions are apparently new.

On the other hand, the Hunter-Saxton equation belongs to a larger family called Dym-type equation in [7, 8, 9],

$$u_{xxt} + 2u_x u_{xx} + u u_{xxx} - 2\kappa u_x = 0, \quad \kappa = \text{constant}.$$

One of these equations is a member of the Dym hierarchy that has been studied by, amongst others, Kruskal [39], Cao [40], Hunter and Zheng [2] and Alber et al. [7, 8].

We remark that although our focus in this paper is on the case  $\kappa = 0$ , all the arguments presented here can be adapted, with no obvious modifications, to study the corresponding equation  $\kappa \neq 0$ . As it is observed that by substituting  $y = \frac{1}{2}u_{xx}$  into  $y = \frac{1}{2}u_{xx} - \frac{1}{2}\kappa$ , then (4.11) represents the Dym hierarchy and (4.12) will become the Dym-type equation. The analysis from Section 5 to Section 8 can extend line by line to the Dym hierarchy. Hence, it is trivial to investigate the algebro-geometric solutions of Dym hierarchy again.

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